Geometric phase for entangled states of two spin-1/2 particles in rotating magnetic field

2003 J. Phys. A: Math. Gen. 36 1149

(http://iopscience.iop.org/0305-4470/36/4/320)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 1149–1157 PII: S0305-4470(03)53734-9

Geometric phase for entangled states of two spin-1/2 particles in rotating magnetic field

D M Tong[1], L C Kwek[1,2] and C H Oh[1]

[1] Department of Physics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260
[2] National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 639798

E-mail: phyohch@nus.edu.sg

Received 17 September 2002, in final form 22 November 2002
Published 15 January 2003
Online at stacks.iop.org/JPhysA/36/1149

Abstract
The geometric phase for states of two spin-1/2 particles in rotating magnetic field is calculated, in particular, the noncyclic and cyclic non-adiabatic phases for the general case are explicitly derived and discussed. We find that the cyclic geometric phase for the entangled state can always be written as a sum of the phases of the two particles respectively; the same cannot be said for the noncyclic phase. We also investigate the geometric phase of mixed state of one particle in a biparticle system, and we find that the geometric phase for one subsystem of an entangled system is always affected by another subsystem of the entangled system.

PACS numbers: 03.65.Vf, 03.65.−w

1. Introduction

In a seminal paper by Berry [1] on the geometric phase in 1983, the adiabatic cyclic evolution of a quantum system was considered. This discovery has prompted a myriad of activities on various aspects of geometric phase in many areas of physics ranging from optical fibres to anyons. Simon [2] was the first to recast the mathematical formalism of Berry phase within the elegant language of differential geometry and fibre bundles. While it is possible to consider Berry phase under adiabatic evolution, the extension to non-adiabatic evolution is nontrivial and was formulated in general terms by Aharonov and Anandan [3]. Samuel and Bhandari [4] generalized the idea further by extending it to noncyclic and non-unitary evolution. Further generalizations and refinements on geometric phase by relaxing the conditions of adiabaticity, unitarity and the cyclic nature of the evolution have since been made [5–13].

Quantum entanglement has been one of the most puzzling aspects of quantum theory [14]. Recently entangled states have been found useful for quantum cryptography, quantum teleportation and quantum computations. Moreover, it was found that one could in principle devise a more robust fault-tolerant computer using the notion of geometric phase in designing a CNOT gate. Since geometric phase depends solely on the geometry of the intrinsic spin space, it is deemed to be less susceptible to noise from the environment.

In a recent paper [10], Sjöqvist calculated the geometric phase for a pair of entangled spins in a time-independent uniform magnetic field. This is an interesting development in holonomic quantum computer since the study of spin systems effectively allows us to contemplate the design of a solid state quantum computer [15]. As a step towards realization of this, we consider in this paper spin pairs in rotating magnetic field generalizing the original formulation of time-independent uniform magnetic field. Two spin-$1/2$ particles in a rotating magnetic field is a common physical system. Naturally, one should be able to recover Sjöqvist's result from our study by setting the rotating components of magnetic field to zero. In addition, we find that the geometric phase of the total entangled biparticle system can be decomposed into a sum of the individual geometric phases of the two particles under the cyclic condition; the same cannot be said for the noncyclic phase. Furthermore, the geometric phase of a particle is different from that as computed from the mixed state of the same particle. This result indicates that the geometric phase for one subsystem of an entangled system is always affected by another subsystem of the entangled system, even though the magnetic field is applied only to the first subsystem while the later subsystem keeps free.

2. Rotating magnetic field

Considering two spin-$1/2$ particles in a rotating magnetic field, the interaction Hamiltonian for the system is

$$\hat{H}(t) = \hat{H}_a(t) + \hat{H}_b(t) \tag{1}$$

where $\hat{H}_\mu(t) = \omega_\mu(\sin\theta\cos\omega t S_{\mu x} + \sin\theta\sin\omega t S_{\mu y} + \cos\theta S_{\mu z})(\mu = a, b)$ and $\vec{S}_\mu$ is the spin operator. The states $|\Psi(t)\rangle$ of the system obey Schrödinger equation,

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = (\hat{H}_a(t)\otimes I + I\otimes\hat{H}_b(t))|\Psi(t)\rangle \tag{2}$$

where I is a 2×2 unit matrix and $|\Psi(t)\rangle$ is a 4×1 matrix. We write the general solution in the following form,

$$\begin{aligned}|\Psi(t)\rangle = {} & c_1|\phi_a(t)\rangle\otimes|\phi_b(t)\rangle + c_2|\phi_a(t)\rangle\otimes|\psi_b(t)\rangle \\ & + c_3|\psi_a(t)\rangle\otimes|\phi_b(t)\rangle + c_4|\psi_a(t)\rangle\otimes|\psi_b(t)\rangle\end{aligned} \tag{3}$$

where

$$|\phi_\mu(t)\rangle = N_\mu\begin{pmatrix}\sin\theta\,\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}t}\\ k_\mu\,\mathrm{e}^{\mathrm{i}\frac{\omega}{2}t}\end{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\omega'_\mu}{2}t} \qquad |\psi_\mu(t)\rangle = N_\mu\begin{pmatrix}k_\mu\,\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}t}\\ -\sin\theta\,\mathrm{e}^{\mathrm{i}\frac{\omega}{2}t}\end{pmatrix}\mathrm{e}^{\mathrm{i}\frac{\omega'_\mu}{2}t}$$

$\omega'_\mu = \sqrt{\omega^2+\omega_\mu^2-2\omega\omega_\mu\cos\theta}$, $k_\mu = \frac{\omega+\omega'_\mu}{\omega_\mu} - \cos\theta$, $N_\mu = \left(k_\mu^2+\sin^2\theta\right)^{\frac{-1}{2}}$ and c_1, c_2, c_3, c_4 are arbitrary constants to be determined by initial conditions. Note that $|\phi_\mu(t)\rangle$ and $|\psi_\mu(t)\rangle$ satisfy the Schrödinger equation for a single particle, and $\langle\phi_\mu(t)|\psi_\mu(t)\rangle = 0$.

For convenience we parametrize the initial state $|\Psi(0)\rangle$ of the system by

$$|\Psi(0)\rangle = r_1\,\mathrm{e}^{\mathrm{i}\alpha_1}|\uparrow\rangle_a\otimes|\uparrow\rangle_b + r_2\,\mathrm{e}^{\mathrm{i}\alpha_2}|\uparrow\rangle_a\otimes|\downarrow\rangle_b + r_3\,\mathrm{e}^{\mathrm{i}\alpha_3}|\downarrow\rangle_a\otimes|\uparrow\rangle_b + r_4\,\mathrm{e}^{\mathrm{i}\alpha_4}|\downarrow\rangle_a\otimes|\downarrow\rangle_b \tag{4}$$

where $r_j, \alpha_j (j = 1, 2, 3, 4)$ are real constants, $|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, and the relation between c_j and $r_j\,\mathrm{e}^{\mathrm{i}\alpha_j}$ is given by

$$\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix} = N\begin{pmatrix}\sin^2\theta & k_b\sin\theta & k_a\sin\theta & k_ak_b\\ k_b\sin\theta & -\sin^2\theta & k_ak_b & -k_a\sin\theta\\ k_a\sin\theta & k_ak_b & -\sin^2\theta & -k_b\sin\theta\\ k_ak_b & -k_a\sin\theta & -k_b\sin\theta & \sin^2\theta\end{pmatrix}\begin{pmatrix}r_1\,\mathrm{e}^{\mathrm{i}\alpha_1}\\ r_2\,\mathrm{e}^{\mathrm{i}\alpha_2}\\ r_3\,\mathrm{e}^{\mathrm{i}\alpha_3}\\ r_4\,\mathrm{e}^{\mathrm{i}\alpha_4}\end{pmatrix}. \tag{5}$$

Here $N = N_aN_b = \left[\left(k_a^2 + \sin^2\theta\right)\left(k_b^2 + \sin^2\theta\right)\right]^{\frac{-1}{2}}$. For normalized states $|\Psi(t)\rangle$, we require $\sum_{j=1}^{4}|c_j|^2 = \sum_{j=1}^{4} r_j^2 = 1$.

Expressions (3),(4) and (5) solve Schrödinger equation (2). These solutions include both entangled states and unentangled states depending on the values of c_j or $r_j\,\mathrm{e}^{\mathrm{i}\alpha_j} (j = 1, 2, 3, 4)$. When $c_1c_4 \neq c_2c_3$, or $r_1\,\mathrm{e}^{\mathrm{i}\alpha_1} \cdot r_4\,\mathrm{e}^{\mathrm{i}\alpha_4} \neq r_2\,\mathrm{e}^{\mathrm{i}\alpha_2} \cdot r_3\,\mathrm{e}^{\mathrm{i}\alpha_3}$, the solutions are entangled states, otherwise, they are unentangled states.

3. Geometric phase for a two-particle system

Geometric phase for a pure state can be obtained from the total phase by subtracting the dynamical phase [5]. Let Φ_G, Φ_T and Φ_D be the geometric phase, total phase and dynamical phase respectively, then

$$\Phi_G = \Phi_T - \Phi_D \tag{6}$$

where

$$\Phi_T = \arg\langle\Psi(0) \mid \Psi(t)\rangle \tag{7}$$

$$\Phi_D = -\mathrm{i}\int_0^t \mathrm{d}t\,\langle\Psi(t)|\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle. \tag{8}$$

To obtain the total phase, we substitute expressions (3) and (5) into (7) and after some lengthy calculations we get

$$\langle\Psi(0)|\Psi(t)\rangle = \cos\frac{\omega_a'}{2}t\cos\frac{\omega_b'}{2}t\left\{\mathrm{i}\left[K_1 + K_2\tan\frac{\omega_a'}{2}t + K_3\tan\frac{\omega_b'}{2}t + K_4\tan\frac{\omega_a'}{2}t\tan\frac{\omega_b'}{2}t\right]\right.$$
$$\left. + \left[K_5 + K_6\tan\frac{\omega_a'}{2}t + K_7\tan\frac{\omega_b'}{2}t + K_8\tan\frac{\omega_a'}{2}t\tan\frac{\omega_b'}{2}t\right]\right\} \tag{9}$$

and

$$\Phi_T = \arctan\frac{K_1 + K_2\tan\frac{\omega_a'}{2}t + K_3\tan\frac{\omega_b'}{2}t + K_4\tan\frac{\omega_a'}{2}t\tan\frac{\omega_b'}{2}t}{K_5 + K_6\tan\frac{\omega_a'}{2}t + K_7\tan\frac{\omega_b'}{2}t + K_8\tan\frac{\omega_a'}{2}t\tan\frac{\omega_b'}{2}t}. \tag{10}$$

Here

$$K_1 = r_{11,44}^{c,-}\sin\omega t$$

$$K_2 = \frac{1}{\omega_a'}\left\{r_{31,42}^{s,+}\omega_a\sin\theta\sin\omega t + \left[r_{11,44}^{c,-}(\omega - \omega_a\cos\theta) - r_{31,42}^{c,+}\omega_a\sin\theta\right]\cos\omega t\right.$$
$$\left. + \left[r_{22,33}^{c,-}(\omega - \omega_a\cos\theta) - r_{31,42}^{c,+}\omega_a\sin\theta\right]\right\}$$

$$K_3 = \frac{1}{\omega_b'}\left\{r_{21,43}^{s,+}\omega_b\sin\theta\sin\omega t + \left[r_{11,44}^{c,-}(\omega - \omega_b\cos\theta) - r_{21,43}^{c,+}\omega_b\sin\theta\right]\cos\omega t\right.$$
$$\left. + \left[r_{33,22}^{c,-}(\omega - \omega_b\cos\theta) - r_{21,43}^{c,+}\omega_b\sin\theta\right]\right\}$$

$$K_4 = \frac{1}{\omega_a'\omega_b'}\big\{\big[r_{44,11}^{c,-}(\omega-\omega_a\cos\theta)(\omega-\omega_b\cos\theta) + r_{21,43}^{c,+}(\omega-\omega_a\cos\theta)\omega_b\sin\theta + r_{31,42}^{c,+}\omega_a\sin\theta(\omega-\omega_b\cos\theta)\big]\sin\omega t + \big[r_{21,43}^{s,+}(\omega-\omega_a\cos\theta)\omega_b\sin\theta + r_{31,42}^{s,+}\omega_a\sin\theta(\omega-\omega_b\cos\theta)\big](\cos\omega t-1)\big\}$$

$$K_5 = r_{11,44}^{c,+}\cos\omega t + r_{22,33}^{c,+}$$

$$K_6 = \frac{1}{\omega_a'}\big\{\big[-r_{11,44}^{c,+}(\omega-\omega_a\cos\theta) + r_{31,42}^{c,-}\omega_a\sin\theta\big]\sin\omega t + r_{31,42}^{s,-}\omega_a\sin\theta(\cos\omega t-1)\big\}$$

$$K_7 = \frac{1}{\omega_b'}\big\{\big[-r_{11,44}^{c,+}(\omega-\omega_b\cos\theta) + r_{21,43}^{c,-}\omega_b\sin\theta\big]\sin\omega t + r_{21,43}^{s,-}\omega_b\sin\theta(\cos\omega t-1)\big\}$$

$$K_8 = \frac{1}{\omega_a'\omega_b'}\big\{\big[r_{43,21}^{s,-}(\omega-\omega_a\cos\theta)\omega_b\sin\theta + r_{42,31}^{s,-}\omega_a\sin\theta(\omega-\omega_b\cos\theta) + r_{41,41}^{s,+}\omega_a\omega_b\sin^2\theta\big]\sin\omega t + \big[-r_{11,44}^{c,+}(\omega-\omega_a\cos\theta)(\omega-\omega_b\cos\theta) + r_{21,43}^{c,-}(\omega-\omega_a\cos\theta)\omega_b\sin\theta + r_{31,42}^{c,-}\omega_a\sin\theta(\omega-\omega_b\cos\theta) - r_{41,41}^{c,+}\omega_a\omega_b\sin^2\theta\big]\cos\omega t + \big[r_{22,33}^{c,+}(\omega-\omega_a\cos\theta)(\omega-\omega_b\cos\theta) + r_{21,43}^{c,-}(\omega-\omega_a\cos\theta)\omega_b\sin\theta + r_{31,42}^{c,-}\omega_a\sin\theta(\omega-\omega_b\cos\theta) - r_{32,32}^{c,+}\omega_a\omega_b\sin^2\theta\big]\big\} \tag{11}$$

where $r_{ij,kl}^{s,\pm} = r_ir_j\sin\alpha_{ij} \pm r_kr_l\sin\alpha_{kl}$, $r_{ij,kl}^{c,\pm} = r_ir_j\cos\alpha_{ij} \pm r_kr_l\cos\alpha_{kl}$, $\alpha_{ij} = \alpha_i - \alpha_j$.

Substituting (3) and (5) into (8), we obtain the dynamical phase Φ_D as

$$\begin{aligned}\Phi_D = {} & \frac{\omega_a(\omega\cos\theta-\omega_a)}{2\omega_a'^2}\big[\big(1-2r_{11,22}^{c,+}\big)(\omega-\omega_a\cos\theta) + 2r_{31,42}^{c,+}\omega_a\sin\theta\big]t \\ & + \frac{\omega_b(\omega\cos\theta-\omega_b)}{2\omega_b'^2}\big[\big(1-2r_{11,33}^{c,+}\big)(\omega-\omega_b\cos\theta) + 2r_{21,43}^{c,+}\omega_b\sin\theta\big]t \\ & + \frac{\omega\omega_a\sin\theta}{4\omega_a'^3}\big[\big(1-2r_{11,22}^{c,+}\big)\omega_a\sin\theta - 2r_{31,42}^{c,+}(\omega-\omega_a\cos\theta)\big]\sin\omega_a't \\ & + \frac{\omega\omega_b\sin\theta}{4\omega_b'^3}\big[\big(1-2r_{11,33}^{c,+}\big)\omega_b\sin\theta - 2r_{21,43}^{c,+}(\omega-\omega_b\cos\theta)\big]\sin\omega_b't \\ & + \frac{\omega\omega_a\sin\theta}{2\omega_a'^2}r_{31,42}^{s,+}(\cos\omega_a't-1) + \frac{\omega\omega_b\sin\theta}{2\omega_b'^2}r_{21,43}^{s,+}(\cos\omega_b't-1)\end{aligned} \tag{12}$$

where the normalization $\sum_{j=1}^{4} r_j^2 = 1$ has been used.

From the total phase equation (10) and the dynamical phase (12), the geometric phase is readily given by equation (6). It is the general result for two spin-1/2 particles in a rotating magnetic field.

4. Remarks

Despite the complexity in the expression for the geometric phase in its full generality, we can extract some interesting results. From equation (12), we find that the dynamic phase can always be separated into two parts corresponding to the phases of the two separate particles. However, the total phase as well as the geometric phase cannot be separated into two parts except under some special cases. The latter observation arises primarily from the entanglement of the two particles. If we allow $\omega_b = 0$, so that only one of the two particles is affected by the rotating magnetic field, we see that the geometric phase is generally different from the phase of the single particle's system. Thus, entanglement essentially affects the geometric phase of a system. In the following, we use the general formulae (6), (10), (12) to discuss some special cases.

4.1. Recovering Sjöqvist's result

If we let $\sin\theta = 0$, the rotating magnetic field in the xy plane vanishes and only the component in direction z remains. The result of spin pairs in time-independent magnetic field then follows. In this case, we have $K_1 = K_4 = K_6 = K_7 = 0$, $K_2 = -\left(1 - 2r_1^2 - 2r_2^2\right)$, $K_3 = -\left(1 - 2r_1^2 - 2r_3^2\right)$, $K_5 = 1$, $K_8 = 1 - 2r_1^2 - 2r_4^2$ giving

$$\Phi_T = -\arctan \frac{\left(1 - 2r_1^2 - 2r_2^2\right)\tan\frac{\omega_a t}{2} + \left(1 - 2r_1^2 - 2r_3^2\right)\tan\frac{\omega_b t}{2}}{1 + \left(1 - 2r_1^2 - 2r_4^2\right)\tan\frac{\omega_a t}{2}\tan\frac{\omega_b t}{2}} \tag{13}$$

$$\Phi_D = -\left[\left(1 - 2r_1^2 - 2r_2^2\right)\frac{\omega_a t}{2} + \left(1 - 2r_1^2 - 2r_3^2\right)\frac{\omega_b t}{2}\right] \tag{14}$$

and

$$\begin{aligned}\Phi_G = -&\left[\arctan \frac{\left(1 - 2r_1^2 - 2r_2^2\right)\tan\frac{\omega_a t}{2} + \left(1 - 2r_1^2 - 2r_3^2\right)\tan\frac{\omega_b t}{2}}{1 + \left(1 - 2r_1^2 - 2r_4^2\right)\tan\frac{\omega_a t}{2}\tan\frac{\omega_b t}{2}}\right] \\ &+ \left[\left(1 - 2r_1^2 - 2r_2^2\right)\frac{\omega_a t}{2} + \left(1 - 2r_1^2 - 2r_3^2\right)\frac{\omega_b t}{2}\right].\end{aligned} \tag{15}$$

Clearly, equation (15) recovers Sjöqvist's result [10]. (Note that our notation differs from Sjöqvist's paper.) One can see this by letting the initial state $|\Psi(0)\rangle$ in equation (4) be $\mathrm{e}^{-\mathrm{i}\beta/2}\cos\frac{\alpha}{2}|n(0)\rangle_a|m(0)\rangle_b + \mathrm{e}^{\mathrm{i}\beta/2}\sin\frac{\alpha}{2}|-n(0)\rangle_a|-m(0)\rangle_b$ (see [10]). A simple change in the notation shows that Φ_G is same as that of [10].

4.2. Geometric phases for some special states

Using equations (10), (12) and (6), we can calculate the phase for *any* initial condition. As an example, we can consider unentangled states and entangled states.

For an unentangled state,

$$|\Psi(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix}_a \otimes \begin{pmatrix}0\\1\end{pmatrix}_b \tag{16}$$

which is obtained by setting $r_1 = r_3 = r_4 = 0, r_2 = 1, \alpha_i = 0\,(i = 1, 2, 3, 4)$. A direct substitution leads to

$$\Phi_T = \arctan \frac{\frac{\omega - \omega_a\cos\theta}{\omega'_a}\tan\frac{\omega'_a}{2}t - \frac{\omega - \omega_b\cos\theta}{\omega'_b}\tan\frac{\omega'_b}{2}t}{1 + \frac{(\omega - \omega_a\cos\theta)(\omega - \omega_b\cos\theta)}{\omega'_a\omega'_b}\tan\frac{\omega'_a}{2}t\tan\frac{\omega'_b}{2}t} \tag{17}$$

and

$$\begin{aligned}\Phi_D = &\left[-\frac{\omega_a(\omega\cos\theta - \omega_a)(\omega - \omega_a\cos\theta)}{2\omega_a'^2} + \frac{\omega_b(\omega\cos\theta - \omega_b)(\omega - \omega_b\cos\theta)}{2\omega_b'^2}\right]\omega t \\ &- \frac{\omega\omega_a^2\sin^2\theta}{4\omega_a'^3}\sin\omega'_a t + \frac{\omega\omega_b^2\sin^2\theta}{4\omega_b'^3}\sin\omega'_b t.\end{aligned} \tag{18}$$

It is obvious that in this case $\Phi_G = \Phi_T - \Phi_D$ can be written as $(\Phi_G)_a + (\Phi_G)_b$. Indeed, the geometric phase of the unentangled state can always be neatly separated into two parts corresponding to the geometric phase for each particle.

For an entangled state, we consider the case in which the initial state is given by a Bell state,

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_a \otimes |\downarrow\rangle_b \pm |\downarrow\rangle_a \otimes |\uparrow\rangle_b]. \tag{19}$$

In this case, $r_1 = r_4 = 0, r_2 = \frac{1}{\sqrt{2}}, r_3 = \pm\frac{1}{\sqrt{2}}, \alpha_i = 0\,(i = 1, 2, 3, 4)$, we have $\Phi_D = 0$ and

$$\langle\Psi(0)|\Psi(t)\rangle = \cos\frac{\omega'_a}{2}t\cos\frac{\omega'_b}{2}t + \frac{(\omega - \omega_a\cos\theta)(\omega - \omega_b\cos\theta) \mp \omega_a\omega_b\sin^2\theta}{\omega'_a\omega'_b}\sin\frac{\omega'_a}{2}t\sin\frac{\omega'_b}{2}t. \tag{20}$$

It follows from the above result that the geometric phase is

$$\Phi_G = \Phi_T = \begin{cases} 0 & \text{if} \quad \langle\Psi(0)\,|\,\Psi(t)\rangle > 0 \\ \text{undefined} & \text{if} \quad \langle\Psi(0)\,|\,\Psi(t)\rangle = 0 \\ \pi & \text{if} \quad \langle\Psi(0)\,|\,\Psi(t)\rangle < 0. \end{cases} \tag{21}$$

Similar results hold for the other two Bell states $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_a \otimes |\downarrow\rangle_b \pm |\downarrow\rangle_a \otimes |\uparrow\rangle_b]$. The phases of these states are consistent with the result of the time-independent field [10].

4.3. Geometric phase for cyclic evolution

Suppose the magnetic field is cyclic, $\hat{H}(\tau) = \hat{H}(0), \tau = \frac{2\pi}{\omega}$, we see that the cyclic condition is realized by

$$|\Psi(\tau)\rangle = \mathrm{e}^{\mathrm{i}\alpha(\tau)}|\Psi(0)\rangle \tag{22}$$

where $\alpha(\tau)$ is real. For arbitrary $\omega, \omega_a, \omega_b$, there are only four unentangled states satisfying the cyclic condition. Setting $t = \frac{2\pi}{\omega}$, we can use the formulae (10) and (12) to calculate the cyclic phases of the four states. Specifically, one of the four states is

$$|\Psi(t)\rangle = |\phi_a(t)\rangle \otimes |\phi_b(t)\rangle \tag{23}$$

which means $r_1\,\mathrm{e}^{\mathrm{i}\alpha_1} = \sin^2\theta, r_2\,\mathrm{e}^{\mathrm{i}\alpha_2} = k_b\sin\theta, r_3\,\mathrm{e}^{\mathrm{i}\alpha_3} = k_a\sin\theta, r_4\,\mathrm{e}^{\mathrm{i}\alpha_4} = k_ak_b$, so that

$$\Phi_T = -\left(\frac{\omega'_a}{\omega} + \frac{\omega'_b}{\omega}\right)\pi \tag{24}$$

$$\Phi_D = \left(\frac{\omega - \omega_a\cos\theta}{\omega'_a} + \frac{\omega - \omega_b\cos\theta}{\omega'_b}\right)\pi - \left(\frac{\omega'_a}{\omega} + \frac{\omega'_b}{\omega}\right)\pi \tag{25}$$

and the geometric phase is

$$\Phi_G = \Phi_T - \Phi_D = -\left(\frac{\omega - \omega_a\cos\theta}{\omega'_a} + \frac{\omega - \omega_b\cos\theta}{\omega'_b}\right)\pi. \tag{26}$$

A similar analysis holds for the other three solutions:

$$|\Psi(t)\rangle = |\phi_a(t)\rangle \otimes |\psi_b(t)\rangle, |\psi_a(t)\rangle \otimes |\phi_b(t)\rangle, |\psi_a(t)\rangle \otimes |\psi_b(t)\rangle \tag{27}$$

and their geometric phases Φ_G are, respectively,

$$\left(-\frac{\omega - \omega_a\cos\theta}{\omega'_a} + \frac{\omega - \omega_b\cos\theta}{\omega'_b}\right)\pi \qquad \left(\frac{\omega - \omega_a\cos\theta}{\omega'_a} - \frac{\omega - \omega_b\cos\theta}{\omega'_b}\right)\pi$$
$$\left(\frac{\omega - \omega_a\cos\theta}{\omega'_a} + \frac{\omega - \omega_b\cos\theta}{\omega'_b}\right)\pi. \tag{28}$$

In this case, the cyclic geometric phase of the two-particle system is just equal to $\frac{\Omega_a}{2} + \frac{\Omega_b}{2}$, where Ω_a and Ω_b correspond, respectively, to half of solid angles enclosed by Bloch vectors $\mathbf{r_a}$, $\mathbf{r_b}$, of the single-particle states.

Now, let us consider the cyclic phase of entangled states. From equation (3), we see that the cyclic condition imposes strong requirements for entangled states in general and there are no solutions unless the oscillating magnetic field is suitably set. In particular, if $\omega'_a = n_a\omega$, $\omega'_b = n_b\omega$, where n_a and n_b are positive integers, then all the entangled states possess cyclic phases. When $t = \tau = \frac{2\pi}{\omega}$, the magnetic field returns to its initial value and the state is

$$|\Psi(\tau)\rangle = \mathrm{e}^{\mathrm{i}\alpha(\tau)}|\Psi(0)\rangle \tag{29}$$

where $\alpha(\tau) = \left(1 - \frac{\omega'_a}{\omega}\right)\pi + \left(1 - \frac{\omega'_b}{\omega}\right)\pi \pmod{2\pi}$ is the total phase. The dynamic phase is given by

$$\Phi_D = \left\{\frac{\omega_a(\cos\theta - \omega_a/\omega)}{\omega_a'^2}\left[\left(1 - 2r^{c,+}_{11,22}\right)(\omega - \omega_a\cos\theta) + 2r^{c,+}_{31,42}\omega_a\sin\theta\right]\right. \\ \left. + \frac{\omega_b(\cos\theta - \omega_b/\omega)}{\omega_b'^2}\left[\left(1 - 2r^{c,+}_{11,33}\right)(\omega - \omega_b\cos\theta) + 2r^{c,+}_{21,43}\omega_b\sin\theta\right]\right\}\pi \tag{30}$$

while the geometric phase can be written as

$$\Phi_G = \Phi_T - \Phi_D = (\Phi_G)_a + (\Phi_G)_b \tag{31}$$

where

$$(\Phi_G)_a = \left(1 - \frac{\omega'_a}{\omega}\right)\pi - \frac{\omega_a(\cos\theta - \omega_a/\omega)}{\omega_a'^2}\left[\left(1 - 2r^{c,+}_{11,22}\right)(\omega - \omega_a\cos\theta) + 2r^{c,+}_{31,42}\omega_a\sin\theta\right]\pi \tag{32}$$

and

$$(\Phi_G)_b = \left(1 - \frac{\omega'_b}{\omega}\right)\pi - \frac{\omega_b(\cos\theta - \omega_b/\omega)}{\omega_b'^2}\left[\left(1 - 2r^{c,+}_{11,33}\right)(\omega - \omega_b\cos\theta) + 2r^{c,+}_{21,43}\omega_b\sin\theta\right]\pi. \tag{33}$$

Note that $(\Phi_G)_a$ and $(\Phi_G)_b$ are the geometric phases corresponding to the particle a and particle b, respectively. Equation (31) shows that the cyclic geometric phases of the two-particle system can always be regarded as two separable parts corresponding to the phases of two single particles despite the fact that the particles are entangled.

The conclusion holds for all other cyclic states. It is also valid even for the general cyclic states, which satisfy $|\Psi(\tau)\rangle = \mathrm{e}^{\mathrm{i}\alpha(\tau)}|\Psi(0)\rangle$ (τ is any positive constant), irrespective of whether $\hat{H}(\tau) = \hat{H}(0)$ holds or not. To prove this, one may directly solve $|\Psi(\tau)\rangle = \mathrm{e}^{\mathrm{i}\alpha(\tau)}|\Psi(0)\rangle$ with equation (3), which is just equivalent to a set of homogeneous equations for c_1, c_2, c_3, c_4. For a nontrivial solution, the coefficient determinant should be zero, from which one may get $\alpha(\tau)$ and find

$$\alpha(\tau) = \alpha_a(\tau) + \alpha_b(\tau) \tag{34}$$

where $\alpha_\mu(\tau) = \arg\left(x_\mu \pm \mathrm{i}\sqrt{1 - x_\mu^2}\right)(\mu = a, b)$ and,

$$x_\mu = \cos\frac{\omega\tau}{2}\cos\frac{\omega'_\mu\tau}{2} + \frac{\omega - \omega_\mu\cos\theta}{\sqrt{\omega^2 + \omega_\mu^2 - 2\omega\omega_\mu\cos\theta}}\sin\frac{\omega\tau}{2}\sin\frac{\omega'_\mu\tau}{2}.$$

Equation (34) shows that the cyclic total phase can be written as a sum of the phases of the two particles respectively; so does the cyclic geometric phase (because the dynamic phase can always be divided into two parts under consideration).

4.4. Geometric phase of mixed state

The above result concerning the cyclic phase of entangled states may provide some insight regarding the geometric phase of mixed states. Certainly, the two-particle state given by equation (3) is a pure state with density matrix $\rho_{ab}(t) = |\Psi(t)\rangle\langle\Psi(t)|$. However, if we trace out states of the second particle, we will get the reduced density matrix ρ_a , corresponding to mixed states of particle a. From the reduced density matrix, we can deduce the geometric phase of a mixed state, $(\Phi_G)_a^M$. It is interesting to see whether $(\Phi_G)_a$ is equal to $(\Phi_G)_a^M$.

Using equation (3) and tracing out the states of particle b, we obtain the reduced density matrix for particle a as

$$\begin{aligned}\rho_a(t) = \mathrm{tr}_b(|\Psi\rangle\langle\Psi|) &= c^+_{11,22}|\phi_a(t)\rangle\langle\phi_a(t)| + c^+_{33,44}|\psi_a(t)\rangle\langle\psi_a(t)| \\ &\quad + c^+_{13,24}|\phi_a(t)\rangle\langle\psi_a(t)| + c^+_{31,42}|\psi_a(t)\rangle\langle\phi_a(t)| \end{aligned} \tag{35}$$

where $c^{\pm}_{ij,kl} = c_i c_j^* \pm c_k c_l^*$. In order to calculate the geometric phase of $\rho_a(t)$, we rewrite it in the following form,

$$\rho_a(t) = \omega_+|\alpha_+(t)\rangle\langle\alpha_+(t)| + \omega_-|\alpha_-(t)\rangle\langle\alpha_-(t)| \tag{36}$$

where

$$|\alpha_\pm(t)\rangle = \frac{1}{\sqrt{\left(\omega_\pm - c^+_{11,22}\right)^2 + \left|c^+_{13,24}\right|^2}}\left[c^+_{13,24}\,|\,\phi_a(t)\rangle + \left(\omega_\pm - c^+_{11,22}\right)|\psi_a(t)\rangle\right]$$

and

$$\omega_\pm = \frac{1 \pm \sqrt{1 - 4|c_1c_4 - c_2c_3|^2}}{2}.$$

Note that $|\alpha_+(t)\rangle$ and $|\alpha_-(t)\rangle$ are two orthonormal pure states. With the above decomposing and using the method given by [8], we arrive at the geometric phase for mixed state $\rho_a(t)$,

$$(\Phi_G)_a^M = \arctan(r \tan\Phi) \tag{37}$$

where

$$\begin{aligned}\Phi &= \left(1 - \frac{\omega_a'}{\omega}\right)\pi - \frac{\omega_a(\cos\theta - \omega_a/\omega)}{r\omega_a'}\left(c^+_{11,22} - c^+_{33,44}\right)\pi \\ &= \left(1 - \frac{\omega_a'}{\omega}\right)\pi - \frac{\omega_a(\cos\theta - \omega_a/\omega)}{r\omega_a'^2}\left[\left(1 - 2r^{c,+}_{11,22}\right)(\omega - \omega_a\cos\theta) \right. \\ &\quad \left. + 2r^{c,+}_{31,42}\omega_a\sin\theta\right]\pi \end{aligned} \tag{38}$$

and $r = \sqrt{1 - 4|c_1c_4 - c_2c_3|^2}$.

By comparing equation (37) with (32), we find that the two geometric phases computed for the particle a are different, i.e. $(\Phi_G)_a^M \neq (\Phi_G)_a$. Only at $r = 1$, when $|\Psi(t)\rangle$ is an unentangled state, the two phases are the same. This conclusion holds even in the case $\hat{H}_b(t) = 0$. We may suppose that two particles (a, b) are initially entangled together, magnetic field is applied only to particle a while particle b is not affected by the magnetic field. The above discussion shows that $(\Phi_G)_a^M$ is still different from $(\Phi_G)_a$. The result then illustrates that the geometric phase for one subsystem of an entangled system is always affected by other subsystem of the entangled system even though the other subsystem is free.

Acknowledgments

We would like to thank Artur Ekert for his beneficial discussion. This work was supported by NUS Research grant no R-144-000-054-112.

References

[1] Berry M V 1984 *Proc. R. Soc.* A **392** 45
[2] Simon B 1983 *Phys. Rev. Lett.* **51** 2167
[3] Aharonov Y and Anandan J 1987 *Phys. Rev. Lett.* **58** 1593
Anandan J and Aharonov Y 1988 *Phys. Rev.* D **37** 1863
[4] Samuel J and Bhandari R 1988 *Phys. Rev. Lett.* **60** 2339
[5] Mukunda N and Simon R 1993 *Ann Phys. NY* **228** 205
[6] Pati A K 1995 *Phys. Lett.* A **202** 40
[7] Wagh A G and Rakhecha V C 1998 *Phys. Rev. Lett.* **81** 1992
[8] Sjöqvist E *et al* 2000 *Phys. Rev. Lett* **85** 2845
[9] Manini Nicola and Pistolesi F 2000 *Phys. Rev. Lett.* **85** 3067
[10] Sjöqvist E *et al* 2000 *Phys. Rev.* A **62** 022109
[11] Zhu S L, Wang Z D and Zhang Y D *Phys. Rev.* B **61** 1142
[12] Wang X B, Kwek L C and Oh C H 2000 *Phys. Rev.* A **62** 032105
Wang X B and Matsumoto K 2001 *Phys. Rev. Lett.* **87** 097901
[13] Barry C *et al* 2001 *Phys. Rev. Lett.* **86** 369
[14] Schrödinger E 1935 Die gegenwärtige Situation in der Quantenmechanik *Naturwissenschaften* **23** 807–49 (Engl. Transl. 1980 The present situation in quantum mechanics *Proc. Am. Phil. Soc.* **124** 323–38; reprinted in *Quantum Theory and Measurement* 1983 ed J A Wheeler and W H Zurek (Princeton, NJ: Princeton University Press) pp 152–67)
[15] Burkard G and Loss D 2002 *Phys. Rev. Lett.* **88** 047903
DiVincenzo D P, Burkard G, Loss D and Sukhorukov E V 1999 *Quantum Mesoscopic Phenomena and Mesoscopic Devices in Microelectronics* ed I O Kulik and R Ellialtioglu (Ankara, Turkey: NATO Advanced Study Institute)